Hypergraph 2-coloring and sharp thresholds for non-uniform structures

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Properties of random hypergraphs are extensively studied in the literature, but previous efforts were predominantly focused on uniform hypergraphs (i.e. with all the edges of the same size). Our study investigates how results for uniform setting transfer to the non-uniform case.

We examined both algorithmic and non-constructive aspects of 2-colorability. In both settings we found sufficient criterions aggregating information about different edge sizes. On the algorithmic side, surprisingly, non-uniformity can be used to improve the natural generalization of the lower bound. We also extended Friedgut's sharp threshold machinery to cover non-uniform random hypergraphs, additionally showing that 2-colorability has a sharp-threshold in the non-uniform setting as well.

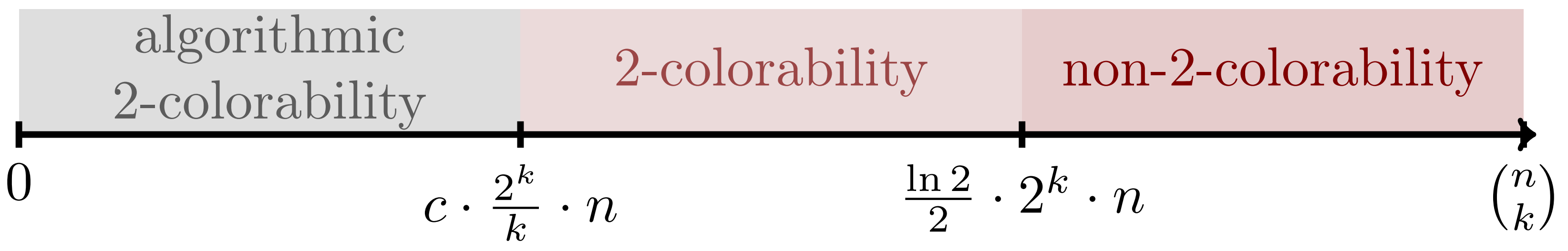
Two-coloring random hypergraphs

Definitions and extension of the standard model.

A hypergraph $\mathcal{H} = (V, E)$ consists of vertices V and edges $E \subseteq \mathcal{P}(V)$. It is 2-colorable if V can be colored red/blue with no monochromatic edge.

The standard model in the literature is $H_k(n, m)$, the hypergraph analogue of $G(n, m)$: on n vertices, it samples exactly m edges of size k uniformly at random.

Known regimes for $H_k(n, m)$



algorithmic 2-colorability

2-colorability

non-2-colorability

0

$c \cdot \frac{2^k}{k} \cdot n$

$\frac{\ln 2}{2} \cdot 2^k \cdot n$

$\binom{n}{k}$

Sources: AKKT [1]: $\Theta(2^k n/k)$; AM [2]: $(\ln 2/2)2^k n$.

We study the non-uniform model $H(n; \mathcal{M})$, where $\mathcal{M}(k)$ specifies how many k-edges are sampled for each arity k.

All statements are asymptotic with $n \to \infty$, and all arities k satisfying $\mathcal{M}(k) > 0$ bounded by some constant.

When is $\mathcal{H} \in H(n; \mathcal{M})$ 2-colorable?

A weighted criterion for 2-colorability.

Shorter edges are more likely to become monochromatic. Thus, in the non-constructive setting, we replace raw edge counts with weighted sum. Let $\mathcal{M}(k) := \lambda_k \cdot 2^k n$.

Two-colorability regime:
If for some $\varepsilon > 0$, $\sum_k \frac{\lambda_k}{1-2^{1-k}} < \frac{\ln 2}{2}(1 - \varepsilon)$. Then for large enough minimal k with $\lambda_k > 0$:

$$\lim_{n\to\infty} \mathcal{P}\left(\mathcal{H} \in H(n; \mathcal{M}) \text{ is 2-colorable}\right) = 1$$

Non-two-colorability regime:
If $\sum_k \left(1 + 2^{-k}\right)\lambda_k > \frac{\ln 2}{2}$, then:

$$\lim_{n\to\infty} \mathcal{P}\left(\mathcal{H} \in H(n; \mathcal{M}) \text{ is 2-colorable}\right) = 0$$

Sharp-threshold for non-uniform structures

Extending Friedgut's result to non-uniform setting.

The second moment method alone only guarantees 2-colorability with positive probability. Upgrading the lower bound above to a.a.s. requires showing that the 2-colorability threshold is sharp.

Friedgut's [4] [3] sharp threshold machinery was originally developed for uniform models. We managed to verify that his proof extends to non-uniforms settings under minor technical assumptions.

Sharp threshold characterization:
Every monotone symmetric property of non-uniform hypergraphs is either sharp, or locally witnessed — i.e. there is a balanced, constant-size hypergraph whose random copy boosts the probability more than uniformly increasing all edge densities by a fixed factor.

AKKT algorithm

Same algorithm, non-uniform analysis.

We revisit the AKKT [1] 2-coloring algorithm for k-uniform hypergraphs. Our refined proof naturally extends to non-uniform setting, allowing to exploit the non-uniformity of our structures.

During the procedure, a j-tail is a partially colored edge with exactly j uncolored vertices left, while already colored vertices in the edge have the same color.

Algorithm 1 AKKT algorithm

```
do $n/2$ times
    if there exists some 3-tail or 2-tail $e$ then
        pick any two uncolored vertices $x, y \in e$ and color $x$ to $RED$ and $y$ to $BLUE$.
    else
        pick any two uncolored vertices $x, y$ and color $x$ to $RED$ and $y$ to $BLUE$.
    end if
end do
```

AKKT sufficient condition:
If in every step of the procedure, the expected number of new 3-tails is bounded by $1 - \varepsilon$, for $\varepsilon > 0$, then a.a.s. the algorithm succeeds.

Exploiting the non-uniformity

How to pack more edges into hypergraph?

AKKT is inherently non-uniform: for a fixed arity k, the expected number of new 3-tails is unimodal in time and peaks near $\left(1 - \frac{3}{k-1}\right) \cdot \frac{n}{2}$-th step. Moreover, larger arities peak later and in narrower windows.

With $\mathcal{M}(k) = c\frac{2^k}{k}n$, the plots below show the expected count of new 3-tails for distributions with single edge size allowed and for a distribution allowing three different arities.

(a) $k = 10$ **(b)** $k = 100$ **(c)** $k_0 = 10, k_1 = 70, k_2 = 300$

Figure 1: Expectation of new 3-tails throughout the execution.

Sufficiently separated arities produce obstructions at almost-disjoint windows. This can be utilized to obtain dsitributions with arbitrary large weighted sum of edges.

Unbounded total weight:
For every $c > 0$, there exists a non-uniform distribution with $\mathcal{M}(k) = \lambda_k \frac{2^k}{k} n$ and $\sum_k \lambda_k > c$ on which AKKT succeeds a.a.s.

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