Thresholds in Erdős–Rényi graphs

Based on works of E. Friedgut

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Erdős–Rényi graphs

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Motivation

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Uncertain interval

Given some property $\mathcal{A}$, e.g. connectivity or appearance of a triangle it is natural to ask how does $\mu_p(\mathcal{A})$ changes with p. We can clearly see that at boundary values of p it's 0 and 1 in both of these cases. It might be interesting to ask *where* does this value changes and how *rapidly*.

Every monotone property has a threshold

Critical interval

Let's fix $\varepsilon > 0$. Then Let p_0, p_1 be s.t. $\mu_{p_0}(\mathcal{A}) = \varepsilon$ and $\mu_{p_1}(\mathcal{A}) = 1 - \varepsilon$.

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Friedgut, 1996

In a paper "Every monotone property has a sharp threshold", Friedgut showed the following. If $\mu_p(\mathcal{A}) > \varepsilon$, then $\mu_q(\mathcal{A}) > 1 - \varepsilon$ for $q = p + c \cdot \frac{\ln(\frac{1}{2\varepsilon})}{\ln n}$. Hence the length of critical interval is at most

$$p_1 - p_0 \leq c \cdot \frac{\ln\left(\frac{1}{2\varepsilon}\right)}{\ln n}$$

where c is an absolute constant.

Sharp threshold

Sharp threshold

Let $\mathcal{A}_n$ be a property of n-vertex graphs and $p^* = p^*(n)$ be such probability, that $\mu_{p^*}(\mathcal{A}) = 1/2$. We say that $\mathcal{A}$ has a sharp threshold iff for any $\varepsilon > 0$:

$$\lim_{n\to\infty} \mu_p\left(\mathcal{A}_n\right) = \begin{cases} 1, & \text{if } p > (1+\varepsilon)\, p^* \\ 0, & \text{if } p < (1-\varepsilon)\, p^* \end{cases} \tag{1}$$

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Coarse threshold

Let's additinally observe, that if $\mathcal{A}$ has a coarse threshold, then there exist $p^* = p^*(n)$ in the critical interval s.t. $p^* \cdot \frac{d\mu_p}{dp} \leq c$.

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As hypergraph are a generalization of graph, their coloring can be generalized analogously:

Hypergraph coloring

A coloring of $\mathcal{H} = (V, E)$ is a function $c : V \to \mathbb{N}$. A coloring is proper if for any $e \in E$, there are some $v, u \in E$ s.t. $c(v) \neq c(u)$.

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We can consider an analogous model to $G(n, p)$, let's call it $H_k(n, p)$ s.t. each (hyper)edge appears independently with probability $p = p(n)$.

Hypergraph 2-coloring

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CSP-like approach

An important subdomain in hypergraph research is 'What is the maximum density of hypergraph s.t. it's 2-colorable with high probability?'. With k fixed and n approaching inifnity, we ask for the maximal $p = p(n)$ s.t. $H_k(n, p)$ is 2-colorable with high probability.

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Convenient notation

Let's observe that the expected number of edges in $H_k(n, p)$ is $\binom{n}{k} \cdot p$. While working with this model, it's often more convenient to parametrize the model by the expected number of edges instead of providing p explicitly.

Hypergraph 2-coloring - known results

Upper bound

In the early 1960' Erdős showed (quite trivial) upper bound using probabilistic method. He showed, that random hypergraphs with the expected number of edges not smaller than $\frac{\ln 2}{2} 2^k n$ are not 2-colorable with high probability (approaching 0 as n approaches infinity).

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Current "best" lower bound

In 2001, Achlioptas and Moore showed (by a quite technical argument) that hypergraphs with average number of edges not exceeding $\frac{\ln 2}{2}\left(2^k - 1\right) n$ is 2-colorable with positive probability.

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How to approach sharp thresholds?

Our examples

We can classify our two examples. It is well known (and we will show it later), that graph connectivity has a sharp threshold. Moreover it's easy to check that property of containing a triangle has a coarse threshold. It's easy to check, that for any $c > 0$ the probability of appearance of triagnle in $G(n, c/n)$ is roughly $1 - e^{-c^3/6}$.

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Intuition behind coarse thresholds

"All monotone graph properties with a coarse threshold may be approximated by a local property."

Friedgut's theorem

Few definitions

Graph is *balanced* if its average degree is no smaller than any of its subgraph.

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Friedgut's main theorem

There exists a function $k(\varepsilon, c)$, such that for all $c > 0$, any n and any monotone symmetric family of graphs $\mathcal{A}$ on n vertices, such that $p \cdot I \leq c$, for every $\varepsilon > 0$ there exists a monotone symmetric family $\mathcal{B}$ such that $||\mathcal{B}|| \leq k(\varepsilon, c)$ and $\mu_p(\mathcal{A} \triangle \mathcal{B}) \leq \varepsilon$. Furthermore the minimal graphs in $\mathcal{B}$ are all balanced.

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Intuition

What this basically means is that almost every graph having property $\mathcal{A}$ has some B_i as a subgraph.

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Let $0 < \alpha < 1$. There exist functions $B(\varepsilon, c), b_1(\varepsilon, c), b_2(\varepsilon, c)$ such that for all $c > 0$, any n and any monotone symmetric family of graphs $\mathcal{A}$ on n vertices such that $p \cdot I \leq c$ and $\alpha < \mu_p(\mathcal{A}) < 1 - \alpha$, for every $\varepsilon > 0$ there exists a graph G with the following properties:

- G is balanced
- $b_1 < E(G) < b_2$
- $|G| \leq B$
- Let $Pr(\mathcal{A}|G)$ denote the probability that a random graph belongs to $\mathcal{A}$ conditioned on the appearance of $\overline{G}$, a specific copy of G. Then $Pr(\mathcal{A}|G) \geq 1 - \varepsilon$.

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If a property doesn't have the threshold probability of the form $p = \Theta\left(n^{\alpha(n)}\right)$, where $\alpha(n)$ is a rational, then it has sharp threshold.

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- Show that $G \notin \mathcal{A}$.
- Show that adding a random $G(n, \varepsilon p)$ edges gives a larger boost to probability of landing in $\mathcal{A}$ than a random copy of G.

Connectivity has sharp threshold

Equipped with our new tools, we can show that graph connectivity is indeed sharp. From our previous remark, we could say that since its threshold probability is $\Theta\left(\frac{\ln n}{n}\right)$ it has a sharp threshold, however this wouldn't be satisfying.

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With this, it's easy to show that adding a copy of $G(n, \varepsilon p)$ will also induce connectivity if $\varepsilon p\binom{n}{2} \to \infty$ which is true for $p = \Theta\left(\frac{\ln n}{n}\right)$.